The equivalence between two definitions of digital surfaces

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Abstract

Digital surfaces deal with properties of surfaces in digital spaces. In the early 80's, researchers began to establish definitions for digital surfaces. Unlike surfaces in continuous spaces, digital surfaces have different characteristics. A general and intuitive definition for digital surfaces is still an open problem. This paper presents a proof of the equivalence of two digital surface definitions. One of the definitions was developed based on simple surface points given by Morgenthaler and Rosenfeld [1], and the second uses a parallel-move concept as given by direct adjacency [2,3]. A by-product of this proof of equivalence is that any simple surface point in a simple surface belongs to one of six types of simple surface points. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

A digital surface usually means the digitized points of a continuous surface. Different sampling ratios can generate different digital surfaces for a continuous surface. On the other hand, for a set of sampled surface points (e.g. from CT), one usually generates a fitted continuous surface to represent the original points. Different fitting algorithms can produce different continuous surfaces. However, such translations are not always necessary. How to deal with a surface in a digital domain has become an important topic. Researchers think that studies of digital curves, surfaces, and solid objects may provide a mathematical foundation for image processing, recognition, and representation [4].

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A digital space is a domain which can be stored in computers. Typically, a one-dimensional digital space means the integer set $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$, and a m-dimensional digital space is $Z^m = Z \times Z... \times Z$. Because computers can only store limited data, we sometimes use $\Sigma_1 = \{1, 2, ..., N\}$ to represent a one-dimensional space. Thus, an m-dimensional digital space is an m-dimensional array Σ_m. We also call Σ_m an m-dimensional grid space. Intuitively, a digital surface is a point set of Σ_m which looks like a surface but does not look like a curve or solid object.

Digital surfaces were first studied by Artzy, Frieder, and Herman in 1981 [5]. They defined a digital surface as the face of a solid object which consisted of unit cubes (called voxels). This is a very intuitive concept, but it is very weak. One can easily generate examples so that they do not satisfy the mathematical definition of surfaces: a neighborhood of each point on a surface is homeomorphic to a 2D disk. In addition, such a surface is always closed. Morgenthaler and Rosenfeld [1] on the other hand, only considered discrete points to form a "digital" surface in Σ_3. Morgenthaler and Rosenfeld gave the first mathematical definition for digital surfaces which is not dependent on continuous surfaces. Unfortunately, Morgenthaler–Rosenfeld's surfaces are not very intuitive and hence are not easy to understand. In order to verify Morgenthaler and Rosenfeld's surfaces are visually reasonable, Kong and Roscoe [6] discussed the continuous analog of Morgenthaler and Rosenfeld's surfaces. In fact, Kong and Roscoe gave an interpolation in three dimensional Euclidean spaces for each type of "simple surface points" in Morgenthaler and Rosenfeld's surfaces. However, all of these can only deal with the "local property" of a point so that they only considered closed surfaces.

In order to assimilate both intuitiveness and strictness in defining a digital surface, we introduced the concept of parallel-moves and gave an intuitive definition for digital surfaces in 3D grid spaces [2,3]. Although some intuitive ideas appeared in Kong–Roscoe [6], the surface definition of [2,3] used a recursive methodology which can be extended to more general cases such as general digital manifolds.

This parallel-move based definition is able to deal with boundary points as well as "inner" points. Using this definition, we have developed several fast algorithms for surface decision, boundary searching, and surface tracking [3]. Furthermore, these algorithms can be modified to the recognizing digital manifolds [2]. Thus, with parallel-move based definition, one can consider both closed and unclosed digital surfaces and design fast algorithms for surface decision, boundary searching, and surface tracking.

However, the parallel-move based surface definition in [2,3] only consider direct adjacency. Allowing indirect adjacency, Morgenthaler and Rosenfeld's surfaces have nine types of digital surfaces. Kong and Roscoe also considered all nine cases [6].

There are two reasons we discuss only direct adjacency in this paper. First, among the total nine types of surfaces, there are only two of them contain non-trivial examples [6,7]. Second, only direct adjacency has the property of unique interpretation. An example with multiple interpretations of indirect adjacency will be shown in Section 2. We also give an example to show that a visually true surface point is not in any of nine types. That is to say, Morgenthaler–Rosenfeld's definition is not complete. Therefore, the case of direct adjacency is the most basic and important one. For indirect adjacency, it is still an open problem to give a simple and intuitive definition of digital surfaces [7].

In this paper, we will prove that Morgenthaler–Rosenfeld's simple surface point in direct adjacency is the regular (inner) surface point defined in [2,3]. Therefore, we show the equivalence between Morgenthaler–Rosenfeld's surfaces and the regular closed surfaces defined in [2,3]. This proof brings interesting result: Morgenthaler–Rosenfeld's simple surface points have exactly six types [8]. The by-product of this paper may be more important to digital topology.

We will introduce the basic concepts and Morgenthaler–Rosenfeld's simple surfaces in Section 2 and the parallel-move based digital surfaces in Section 3. Then, we will prove the equivalence between these two definitions of digital surfaces in Section 4.

2. Simple digital surfaces in Σ_3

Digital surfaces were first studied by Artzy et al. [5]. However, the first mathematical definition of closed digital surfaces in discrete spaces was given by Morgenthaler and Rosenfeld [1]. Based on this definition, many important theoretical results were obtained, e.g., the discrete Jordan theorem [1,4]. An alternative approach was introduced by Kovalevsky [9]. In his work, digital images were represented by topological spaces. Recently, Rosenfeld, Kong and Wu have examined and proven some basic properties of digital surfaces [10].

To begin with, we define the concept of point adjacency. Let us first consider a two-dimensional digital space Σ_2. A point $p = (x, y)$ in Σ_2 has two horizontal $(x, y \pm 1)$ and two vertical neighbors $(x \pm 1, y)$. These four neighbors are called directly adjacent points of p. p also has four diagonal neighbors: $(x \pm 1, y \pm 1)$. These eight (horizontal, vertical and diagonal) neighbors are called indirectly adjacent points of p. Here, direct adjacency is a part of indirect adjacency. In 2D, directly adjacent points are called 4-adjacent points, and indirectly adjacent points are called 8-adjacent points.

Let Σ_3 be a 3-dimensional digital space. Two points $p = (x_1, x_2, x_3)$ and $q = (y_1, y_2, y_3)$ in Σ_3 are directly adjacent points if

$$\sum_{i=1}^{3} |x_i - y_i| = 1.$$

Directly adjacent points are also called 6-adjacent points because the number of directly adjacent points to p are 6 in Σ_3. Note, we always assume p is not at the border of Σ_3. p and q are 18-adjacent points if

$$\max_{1 \leqslant i \leqslant 3} |x_i - y_i| = 1,$$

and

$$\sum_{i=1}^{3} |x_i - y_i| \leqslant 2.$$

p and q are 26-adjacent points if

$$\max_{1 \leqslant i \leqslant 3} |x_i - y_i| = 1,$$

and

$$\sum_{i=1}^{3} |x_i - y_i| \leqslant 3.$$

In a three-dimensional space Σ_3, a point has six 6-adjacent points, eighteen 18-adjacent points, twenty six 26-adjacent points. 6-, 18-, 26-adjacent is denoted as α-adjacent, where $\alpha = 6, 18, 26$. The term "indirect adjacency" means all three types of adjacency are allowed.

A simple α-path is a point set $v_0, v_1, \ldots, v_n$, where v_i and v_{i+1} are α-adjacent for all $i = 0, \ldots, n-1$ and $v_i \neq v_j$ for all i, j. Two points p, q are α-connected if there is a simple α-path $v_0, v_1, \ldots, v_n$ such that $v_0 = p$ and $v_n = q$. Suppose that S is a subset of Σ_3, we say S is α-connected or an α-component if any two points in S are α-connected, i.e. there is an α-path in S linked with these two points.

Let $\bar{S} = \Sigma_3 - S = \{a \mid a \in \Sigma_3 \text{ and } a \notin S\}$, and $N_{27}(p)$ be the $3 \times 3 \times 3$ neighborhood of a point p, i.e. $N_{27}(p) = \{a \mid a \text{ is 26-adjacent to } p\}$.

Definition 2.1 (*Morgenthaler–Rosenfeld's surfaces*)[1]. Let $S \subset \Sigma_3$ and $p \in S$, p is called an (α, β)-simple (digital) surface point, $\alpha, \beta = 6, 18, 26$, if
1. $S \cap N_{27}(p)$ has exactly one α-component α-adjacent to p, denote this component A_p.
2. $\bar{S} \cap N_{27}(p)$ has exactly two β-components, c_1 and c_2, β-adjacent to p.
3. If $q \in S$ and q is α-adjacent to p, then q is β-adjacent to both c_1 and c_2.
S is called a (α, β)-simple (digital) surface if every point in S is a (α, β)-simple surface point.

Hence, there are totally nine types of simple (digital) surfaces. According to this definition, a Morgenthaler–Rosenfeld's surface is a simple closed surface because no boundary is defined in this definition. The definition of Morgenthaler–Rosenfeld surfaces focuses on each individual point and its neighbor-

hood. This definition provides a mathematical description for digital surfaces. However, it is difficult to understand the real meaning of the three conditions in the definition. In order to present a visual interpretation of Morgenthaler–Rosenfeld's simple surface points, Kong and Roscoe "transform (hard) digital topology problems into (fairly easy) problems of continuous topology, which are able to solve"[6],

Kong and Roscoe considered continuous surface-unit-cells in continuous space R^3 which are bounded by points in Σ_3 [6]. Note, for raster images Herman and Weber presented the concept of surface-unit-cells in 1982 [11]. A concept of "plates" such as a surface-unit-cell and "plate cycle" such as all plates around a point in $N_{27}(p)$ provide an intuitive interpretation of a simple surface-point. However, the "plates" and "plate cycle" were defined in a very difficult way. Kong and Roscoe considered all nine types of (α, β)-surface points and gave a continuous interpolation for each type [6]. Kong and Roscoe's work is also important theoretically. With a simple supplement to Kong and Roscoe's work, the nine types of surface points can be reduced to only two types: $(6, 26)$- and $(18, 6)$-simple surface points [7]. The remaining digital surfaces only contain trivial examples [6,7]. However, Kong and Roscoe's work is also difficult to understand since they introduced many new topological concepts.

$(6, 26)$- and $(18, 6)$-simple surfaces are the only useful cases. The example in Fig. 1 will show that the $(18,6)$-simple surface has no unique interpretations.

Kong and Roscoe [6] forced such cases into a unique interpretation. That is to say Morgenthaler–Rosenfeld's surfaces are reasonable, but do not give a complete description. Since the ambiguity here is real existence, so for different cases, one should have different interpretations. In Morgenthaler–Rosenfeld's surfaces, only $(6,26)$-surface has the property of consistency.

In addition, the definition of Morgenthaler-Rosenfeld's surfaces (Definition 2.1) is also not visually complete. Let us look at the following example (Fig. 2), point p is surely a surface-point in human interpretation, but it cannot be a (α, β)-simple surface point.

Although, to allow indirect adjacency in real image processing is necessary, to give a solid mathematical analysis is not simple. One reason for this difficulty is the indirect adjacency break Jordan property "a closed curve separates the rest of the plane into two components." Discussion of indirect adjacency is given in [7].

In this paper, we only consider direct adjacency. In what follows, we omit the word direct, that is, adjacency means direct adjacency. We also simplify Definition 2.1 to the following form:

Definition 2.1′ (*Morgenthaler–Rosenfeld's surfaces*)[1]. Let $S \subset \Sigma_3$ and $p \in S$, p is called a simple surface point, if

1. $S \cap N_{27}(p)$ has exactly one 6-component adjacent to p, denote this component A_p.
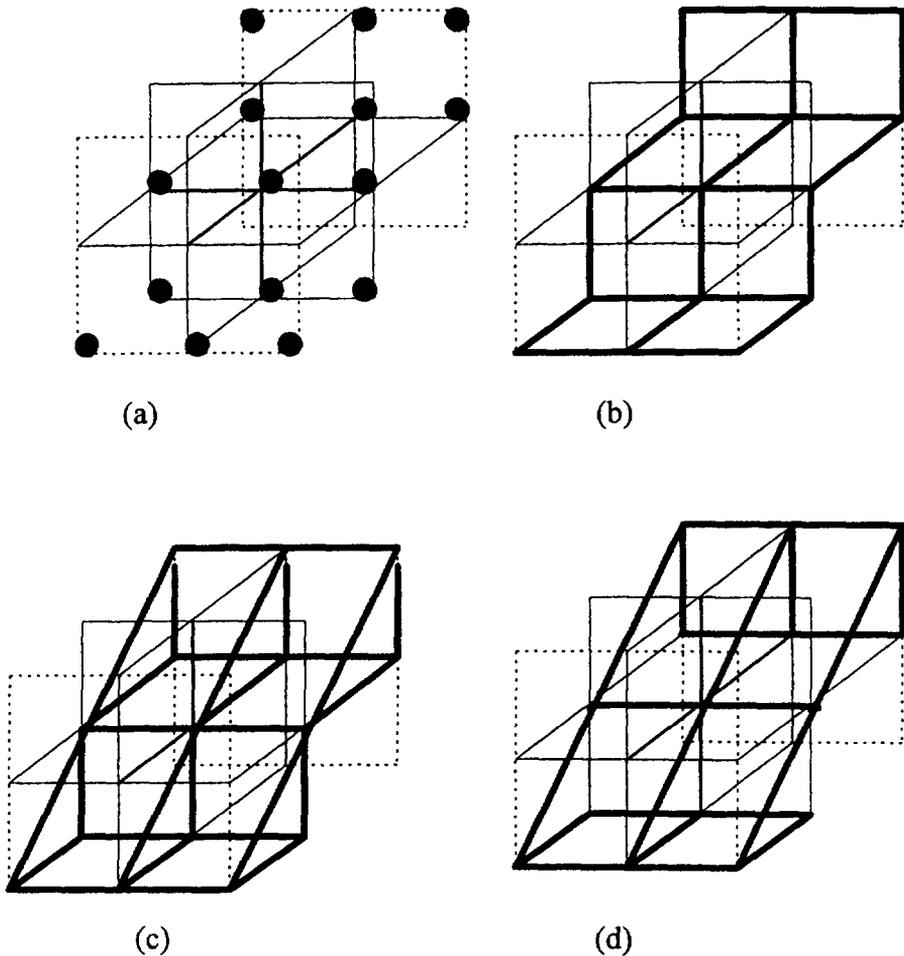
Fig. 1. (a) A digital point set, (b) first interpretation, (c) second interpretation which is a solid object and (d) third interpretation.

2. $\bar{S} \cap N_{27}(p)$ has exactly two 26-components, c_1 and c_2, 26-adjacent to p.
3. If $q \in S$ and q is 6-adjacent to p, then q is 26-adjacent to both c_1 and c_2.

3. Parallel-move based surfaces in Σ_3

Without making use of continuous space, Chen and Zhang [2,3] gave a recursive definition for k-dimensional unit cells using parallel-moves in direct adjacency. Based on line-unit-cells, surface-unit-cells and 3D-unit-cells, an
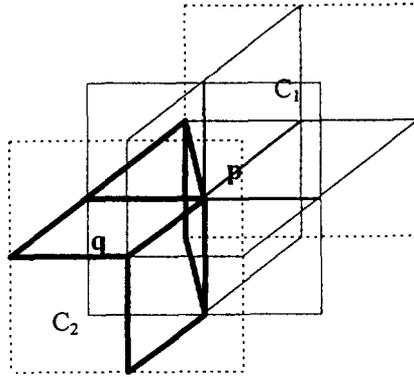
Fig. 2. A visually true surface point but not a (α, β)-surface point for any α and β.

intuitive definition of surfaces was proposed. An optimal surface tracking algorithm was designed by parallelly moving a line-unit-cell. This technique philosophically matches the idea of "curves made by dots and surfaces made by curves."

A point $p = (x, y, z)$ in Σ_3 is called a point-unit-cell (or point-cell). A pair of points $p = (x_1, x_2, x_3)$ and $q = (y_1, y_2, y_3)$ in Σ_3 is called a line-unit-cell (or line-cell), if p and q are adjacent points, i.e.,

$$d(p, q) = \sum_{i=1}^{3} |x_i - y_i| = 1.$$

A line-cell is denoted as a set of p and q, $\{p\} \cup \{q\}$ or $\{p, q\}$.

A surface-unit-cell (or surface-cell) is a set of 4 points which form a unit square parallel to coordinate planes. So, a surface-cell is a union of two line-cells $\{p, p'\}$ and $\{q, q'\}$. The set of $\{p, p', q, q'\}$ are four distinct points, satisfying:

$$d(p, p') = 1 \quad \text{and} \quad d(q, q') = 1,$$

$$d(p, q) = 1 \quad \text{and} \quad d(p', q') = 1.$$

A 3D-unit-cell (or 3D-cell) is a unit cube which includes 8 points. A 3D-cell is the union of two surface-cells $\{p_1, p_2, p_3, p_4\}$ and $\{q_1, q_2, q_3, q_4\}$ such that $\{p_1, p_2, p_3, p_4\} \cap \{q_1, q_2, q_3, q_4\} = \emptyset$ and for all $i = 1, 2, 3, 4$, $d(p_i, q_i) = 1$.

In this note, each line-cell, surface-cell, or 3D-cell is a set of points in Σ_3. We sometimes call a point p a point-cell denoted $\{p\}$. All the operations of sets can be used for line-cells, surface-cells, and 3D-cells. Fig. 3(a)–(c) shows a line-cell, a surface-cell and a 3D-cell, respectively. Some general considerations of unit-cells are given in [2,12].
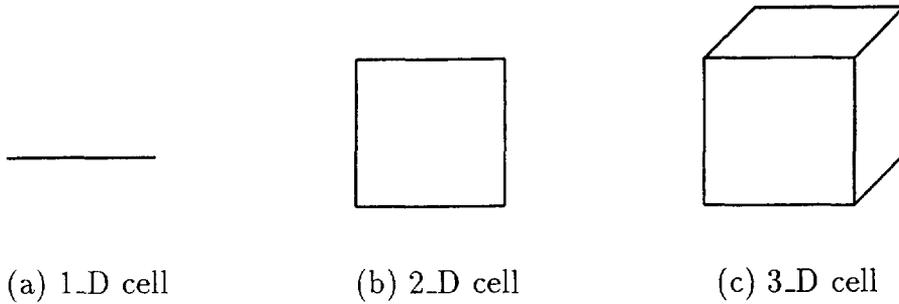
(a) 1_D cell   (b) 2_D cell   (c) 3_D cell

Fig. 3. Three kinds of unit-cells.

Definition 3.1. A line-cell $\{q, q'\}$ is called a parallel-move of line-cell $\{p, p'\}$ if $\{p, p'\} \cap \{q, q'\} = \emptyset$, and $d(p, q) = 1$ and $d(p', q') = 1$.

In the same way, we can define parallel-moves for point-cells, surface-cells, etc.

Lemma 3.1. *In Σ_3, each line-cell has four parallel-moves, and a line-cell and each of its four parallel-moves form a surface-cell.*

In fact, a point-cell and one of its parallel-moves forms a line-cell, and a surface-cell and one of its parallel-moves forms a 3D-cell. We only use the parallel-moves of a line-cell in this paper.

Two line-cells (or surface-cells) are point-adjacent if they share a point. For example, line-cells $l1$ and $l2$ are point-adjacent in Fig. 4(a), and surface-cells $s1$ and $s2$ in Fig. 4(b) are point-adjacent. Two surface-cells are line-adjacent if they share a line-cell. For example, surface-cells $s1$ and $s3$ in Fig. 4(c) are line-adjacent. In order to keep the concept in a simple form, we do not introduce a general definition of unit-cell adjacency here, see [2] for a general definition.

Two line-cells are point-connected if they are two end elements of a line-cell path with point-adjacency. For example, line-cells $l1$ and $l4$ in Fig. 4(a) are point-connected. Two surface-unit-cells are line-connected if they are two end elements of a surface-cell path with line-adjacency. For example, $s1$ and $s2$ in Fig. 4(c) are line-connected.

We now introduce our digital surface definition. We call such a surface a parallel-move based (digital) surface to distinguish it from a Morgenthaler–Rosenfeld's simple surface of Definition 2.1.

Definition 3.2[2,3]. A (point-)connected set $S \subset \Sigma_3$ is a parallel-move based digital surface if
1. Each line-cell of S has at least 1 and at most 2 parallel-moves in S.
2. Any two surface-cells of S are line-connected in S.
3. S does not contain any 3D-cell.

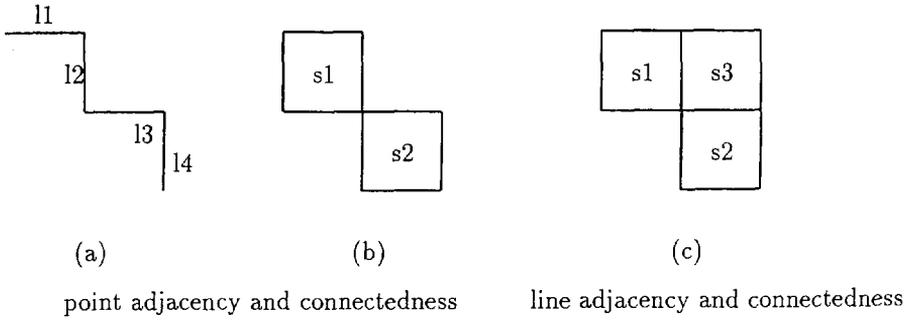point adjacency and connectedness   line adjacency and connectedness

Fig. 4. Adjacency and connectedness.

All of the three conditions in Definition 3.2 are necessary conditions. If the first condition is not satisfied, there is one line-cell in S which has 3 parallel-moves, then S cannot be a surface, (see Fig. 5(a)). If the second condition is violated, that is, there exist two surface-cells in S that are not line-connected, then there must exist two surface-cells in S which are point-connected only (see Fig. 5(b)). If the last condition is not satisfied, S contains some 3D-cell; hence, S cannot be a surface.

Definition 3.3. p is an inner point in surface S if every line-cell containing p has exactly 2 parallel-moves in S. p is a boundary point in S if there exists a line-cell containing p that has only one parallel-move in S. The set of all boundary points is denoted by ∂S, (see Fig. 6).

The differences between Morgenthaler–Rosenfeld's simple surfaces definition and this definition of parallel-move based surfaces are: the former definition is dependent on the point and its neighborhood, but the latter definition is based on a line-cell and its parallel-moves so it can easily deal with boundary points. The Kong and Roscoe's analog is still restricted in the neighborhood of
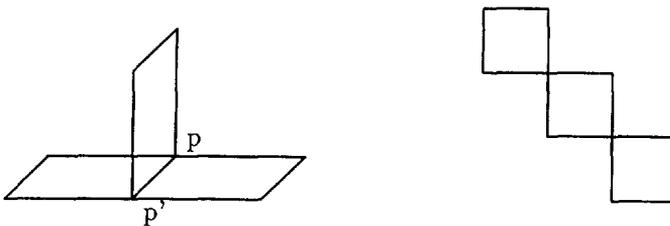
(a) (p,p') has three parallel- moves   (b) Not line-connected's surface-cells

Fig. 5. Two instances which are not considered in surfaces.

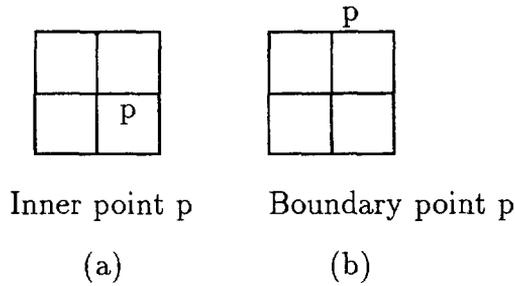Inner point p　　　Boundary point p

(a)　　　　　　　(b)

Fig. 6. Inner and boundary point.

a point [6]. They gave an intuitive interpretation but did not simplify the definition. This analog may make surface recognition procedures even more complicated [7]. In addition, because of the restriction of the neighborhood $N_{27}(p)$, both Morgenthaler–Rosenfeld surfaces and Kong and Roscoe's analog cannot deal with the boundary points of a surface.

4. Simple surface points and regular inner surface points

The definition of parallel-move based surfaces is simple and intuitive. The question is "what is the relationship between such a digital surface and a Morgenthaler–Rosenfeld surface?" This section will show that a closed regular surface is precisely a Morgenthaler–Rosenfeld simple surface. In order to establish the relationship between Morgenthaler–Rosenfeld's simple surfaces and our parallel-move based surfaces, we need to introduce the concept of regular surface points.

If p is a point of a parallel-move based surface S, p is regular if all of S's surface-cells including p are line-connected in S. If p is both inner and regular, then p is called a regular inner surface point. Although most surface points are regular, we show two examples for non-regular surface points in Fig. 7.

To deal with general cases, we can expand the meaning of a regular surface point to any (point-)connected set as follows.

Definition 4.1. Let S be a connected subset of Σ_3. Assume $p \in S$ and $S(p) = S \cap (N_{27}(p) \cup \{p\})$. p is called a regular surface point of S if:
1. Each line-cell in S containing p has at least 1 and at most 2 parallel-moves in $S(p)$.
2. Any two surface-cells containing p in S are line-connected in $S(p)$.
3. $S(p)$ does not contain any 3D-cell.
We say p is a regular inner surface point if p is a regular surface point and each line-cell containing p has exactly two parallel-moves in $S(p)$.
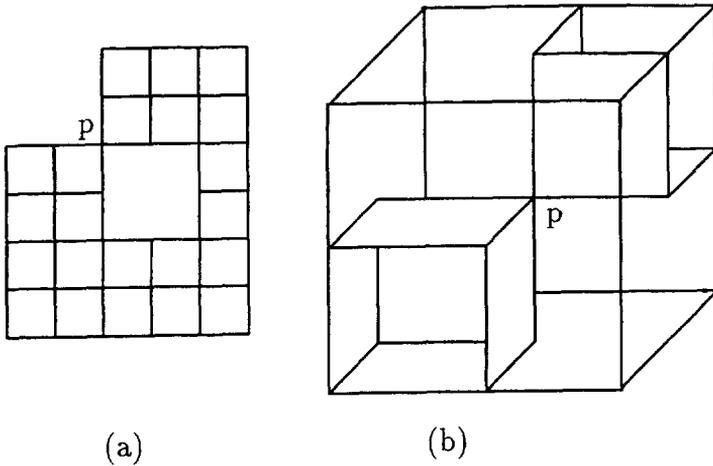
Fig. 7. (a) Non-regular point p; (b) Inner but non-regular surface point p.

Theorem 4.1. *A simple surface point of a set $S \subset \Sigma_3$ is a regular inner surface point, and vice versa.*

Proof. To simplify the proof, we separate it into two parts. In the first part we prove that if p is a simple surface point, then p is also a regular inner surface point. In the second part we prove that if p is a regular inner surface point, then p is a simple surface point.

Proof of part 1. Suppose that p is a simple surface point as reviewed in Section 2. We want to show the following:

1. $S(p) = S \cap (N_{27}(p) \cup \{p\})$ does not have any 3D-cell,
2. each line-cell containing p in S has exactly two parallel moves $S(p)$, i.e. p is inner, and
3. any two surface-cells containing p in S are line-connected in $S(p)$.

First, suppose p is a simple surface point in S. Obviously, p cannot be a corner point of any 3D-cell in S, hence we establish statement 1.

For 2, to begin with, p has three or more (directly) adjacent points in $S \cap N_{27}(p)$. Otherwise, $\bar{S} \cap N_{27}(p)$ is connected, so p is not a simple surface point in accordance with condition (2) of its definition in Section 2.

We define here a grid plane is a set of all points with a fixed z $P_z = \{(x,y,z) | (x,y,z) \in \Sigma_3\}$, all points with a fixed y $P_y = \{(x,y,z) | (x,y,z) \in \Sigma_3\}$, or all points with a fixed x $P_x = \{(x,y,z) | (x,y,z) \in \Sigma_3\}$.

Next, let $p' \in S$ be an arbitrary adjacent point of p. Therefore, there is a grid plane (such as plane1, plane2 or plane3 in Fig. 8) which contains p in $N_{27}(p)$ and does not contain p'. Thus, all directly and indirectly adjacent points of p' are in one side of the grid plane (including the plane). Dependent upon the third condition of the definition of simple surface points, each $c_1(p)$ and $c_2(p)$,
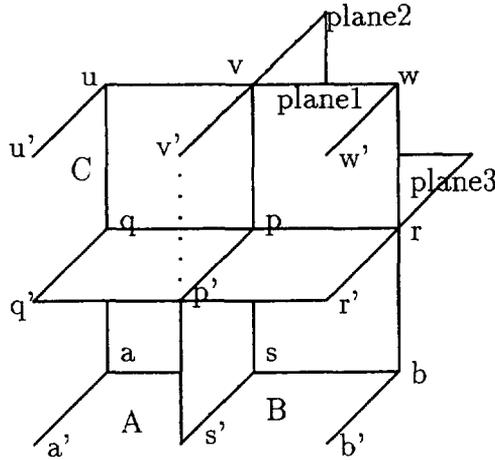
Fig. 8. S in $N_{27}(p)$.

which are defined in Section 2, have one point in the side of the plane because they must be indirectly adjacent to p'. Also, all of the parallel-moves of line-cell $\{p, p'\}$ are in the side of the plane.

If $\{p, p'\}$ has no two parallel-moves in S, then all of the points which are in $\bar{S} \cap N_{27}(p)$ and are in the side of the plane are connected; thus, $c_1(p)$ and $c_2(p)$ are connected. So, $\{p, p'\}$ has two or more parallel-moves.

Now, we prove line-cell $\{p, p'\}$ has no more than two parallel-moves. In contrast, suppose $\{p, p'\}$ has three parallel-moves; without loss of generality, we let the three parallel-moves $\{q, q'\}$, $\{r, r'\}$, and $\{s, s'\}$ of $\{p, p'\}$ be described in Fig. 8.

Suppose we are sure that $\{p, p', q, q', r, r', s, s'\}$ are in S. We let A be the set whose elements are in $\{a, a'\}$ but not in S, and B be the set whose elements are in $\{b, b'\}$ but not in S. Because $S \cap N_{27}(p)$ does not have any 3D-cells, A and B are not empty.

Since p is a simple surface point; A and B, A and C, or B and C must be indirectly connected according to the second condition of the definition of simple surface points in Section 2. Using the same reasoning for A and C, and B and C, we only need to prove: p is no longer a simple surface point when A and B, or A and C are indirectly connected.

(i) If A and B are indirectly connected, then $a \in A$ and $b \in B$; otherwise, A does not connect with B in $N_{27}(p) \cup p$. We know that a and b are indirectly connected. Meanwhile, every s's indirectly adjacent point in $N_{27}(p) \cup p$ is below plane3, and two of them are contained in $c_1(p)$ and $c_2(p)$, respectively. Thus, all of s's indirectly adjacent points must be indirectly adjacent to a or b. Then $c_1(p)$ and $c_2(p)$ defined in Section 2 are indirectly connected, so p is not a simple surface point.

(ii) If A and C are indirectly connected, a must belong to A. We may suppose that $A \cup C \subset c_1(p)$ and $B \subset c_2(p)$. We now must discuss the following two cases.

(ii.a) If $b \in S$ then $c_2(p) = b'$. Thus, q cannot be indirectly adjacent to b, i.e. $c_2(p)$. According to the third condition of the simple surface points' definition, p is not a simple surface point.

(ii.b) If b is not in S, i.e. $b \in c_2(p)$; then we can see if q connects to b and r connects to a, they must pass the plane 2. On the other hand, if $\{u, v, w\} \subset S$, then there is a point of $c_1(p)$ in $\{u', v', w'\}$ based on no 3D-cell in S. However, the point and A cannot be indirectly connected in $N_{27}(p)$, that is, A and C are not indirectly connected. Thus, there must be a point δ in u, v, and w such that it is in $c_1(p) \cup c_2(p)$. If δ is in $c_1(p)$ then each point in plane2 indirectly connects with a or δ. q cannot indirectly connect with the point b. If δ is in $c_2(p)$ then r cannot indirectly connect with the point a. According to the third condition of the simple surface point definition, p is not a simple surface point.

Therefore, we have proven statement 2. We now prove statement 3 to complete part one of the proof. Statement 3 says that any two surface-cells of $S(p)$ are line-connected if p is a simple surface point.

Actually, there are only two cases for two surface-cells A and B including p in $N_{27}(p) \cup p$ which are not line-adjacent (See Fig. 9). In the following we show that these two surface-cells A and B are line-connected in $N_{27}(p) \cup p$ when p is a simple surface point and both A and B are in S.

For case (a), if one of u, v, or w is in S, then surface-cells A and B are line-connected. Otherwise, because each (p, a) and (p, b) have two parallel-moves, C and D must be in S. By the same reasoning, points e and f are in S, so we can see that the p is not a simple surface point because $\bar{S} \cap N_{27}(p)$ has three indirectly connected parts.

For case (b), if u and v are in S, then A and B are line-connected. Otherwise, C and D, or $\{a, e, x, p\}$ and $\{a, p, y, w\}$ must be in S; so we have a case like case (a). Thus, p is not a simple surface point if A and B are not line-connected.

We have thus proven statement 3. To summarize, p is a regular inner surface point if p is a simple surface point.

Proof of part 2. Suppose p is a regular inner surface point. We want to show that p is a simple surface point, i.e. the following three statements are true:

1. $S \cap N_{27}(p)$ has exactly one component adjacent to p, denote this component A_p.
2. $\bar{S} \cap N_{27}(p)$ has exactly two 26-connected components, c_1 and c_2, 26-adjacent to p.
3. If $q \in S$ and q is adjacent to p, then q is 26-adjacent to both C_1 and c_2.

Our strategy for proving part two is different from part one's proof. We just enumerate all possibilities in which a regular inner surface point p can appear, and we then prove that all possible regular inner surface points are simple surface points.
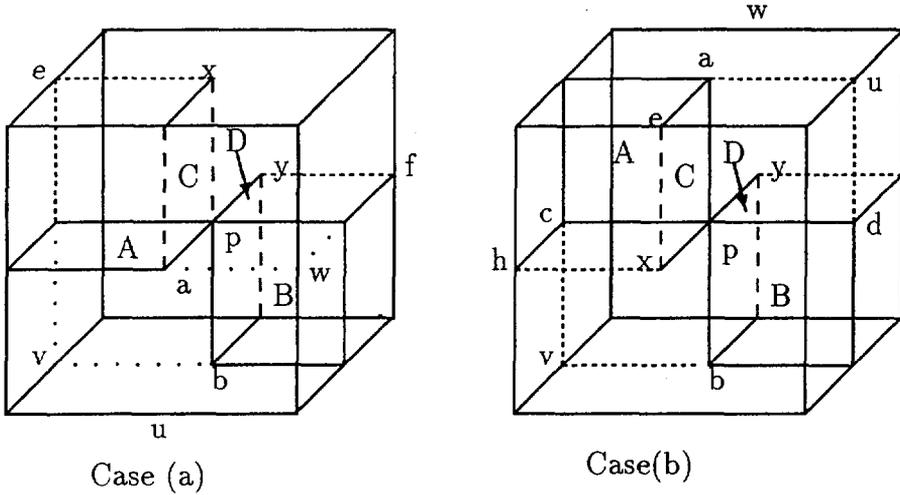
Fig. 9. Two cases in which surface-cells are not line-adjacent.

We know if p is a regular surface point of S, then p is not a corner point of any 3D-cell which is in S. Because of the (point-) connectivity of S, p has a adjacent point denoted by p'. Also, $\{p, p'\}$ has two parallel-moves, denoted by $\{a, a'\}$ and $\{b, b'\}$; therefore, both a and b are also adjacent to p.

Suppose the surface-cell that is formed by $\{p, a\}$ and $\{p, b\}$ is in S, then there are three surface-cells that are line-connected to each other in $S(p)$. Therefore, they can be illustrated as shown in Fig. 10.
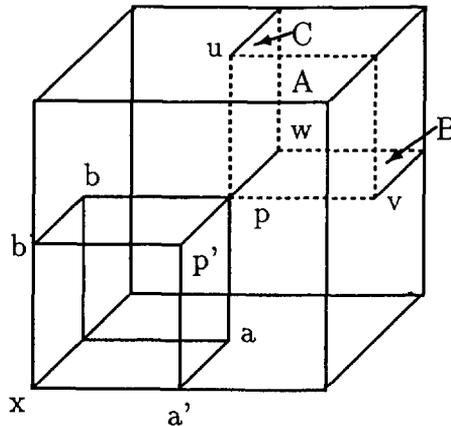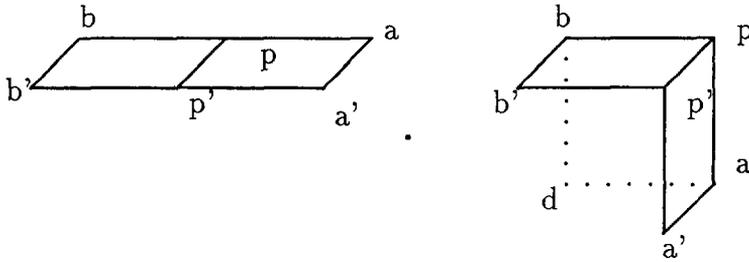

Fig. 10. Not a regular surface point.

Fig. 11. Two cases.

Here x must not be in S as p is not a corner of any 3D-cell. If there is a point u, v, or w that is adjacent to p; then A, B or C must be in S because any line-cell has exactly two parallel-moves. In this case, there exist two surface-cells which are not connected in $N_{27}(p) \cup \{p\}$, so p is not a regular surface point. Thus, there is no such point u, v, or w which is adjacent to p, so we can see p satisfies the definition of a simple surface point.

On the other hand, if the surface-cell formed by $\{p,a\}$ and $\{p,b\}$ is not in S. We can derive only two different cases as shown in Fig. 11.

We now illustrate all possible developments based on the above cases, meaning that p is kept as a regular inner surface point. We only consider the points that are adjacent to p and the surface-cells that contain p. It is straightforward to see that there is only one possible case for a regular inner surface point with exactly three surface-cells.

(i) If there are exactly four surface-cells in $S(p)$, we have only the following two possible cases keeping p as a regular inner surface point (Fig. 12). We can see that p in either (a) or (b) is a simple surface point.

(ii) If there are five or more surface-cells in $S(p) = S \cap (N_{27}(p) \cup p)$, then only two cases which contain three surface-cells can be developed to generate different results. The two cases are given in Fig. 13.
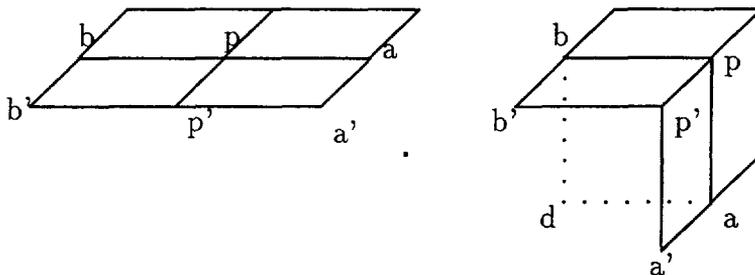
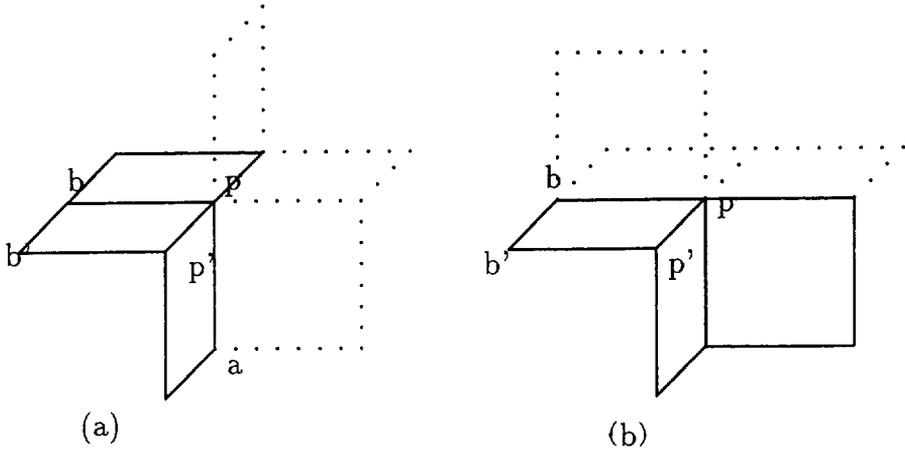Fig. 12. $S(p)$ has exactly four surface-cells.

Fig. 13. Two cases that can be developed.

Each (a) and (b) in Fig. 13 has three possible means to add one more surface-cell, some of which overlap. We can reduce such cases to four distinct cases as follows (Fig. 14).

Next, we continuously add new surface-cells to the above cases and maintain p as a regular inner surface point; so that we arrive at the following seven cases (Fig. 15). We can see that (e) already arrives at the final state, where it has only five surface-cells in $S(p)$. On the other hand, (c) also arrives at the final state because it cannot be a regular inner surface point.

Next, We can see that there is only one choice to add a possible surface-cell for each case in Fig. 15 except (c) and (e). After adding a surface-cell, (a), (b), (d), (f), and (g) can be reduced to two cases. Each of which have six surface-cells including p, where p is a regular inner surface point. We show the two cases as follows (Fig. 16).

Finally, we now show why there are no seven or more surface-cells including p in $S \cap (N_{27}(p) \cup \{p\})$ when p is a regular surface point in S. We know p has six adjacent points in $N_{27}(p) \cup \{p\}$; in other words, there are six line-cells including p. If a surface-cell A including p is in S, then A contains two of the six line-cells. When S has seven surface-cells, there must exist a line-cell which is included by three surface-cells. Therefore, S is not a surface.

From the preceding 5-step process, we obtain three types of regular inner surface points contained by five or six surface-cells. Considering Figs. 11 and 12, we have one regular inner surface point with three surface-cells and two regular inner surface points with exactly four surface-cells. There are only six possibilities for p to be a regular inner surface point [6]. We can see that all of the three kinds of the regular inner surface points satisfy the definition of the simple surface points.
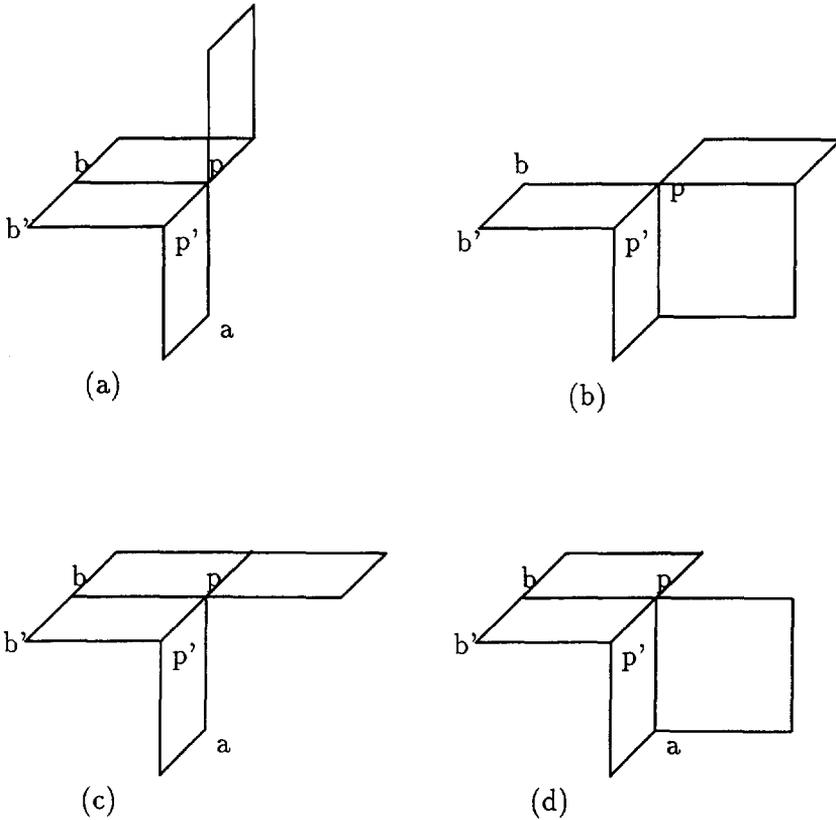
Fig. 14. Four cases.

We now have completed the proof of part 2. Therefore, every regular inner surface point is a simple surface point. Since none of the discussions used the first condition of Morgenthaler–Rosenfeld's definition,

Corollary 4.1. *The second and third conditions of the definition of Morgenthaler–Rosenfeld's surface imply the first condition.*

5. More about regular surface points

In Section 4, we define a regular surface point based on parallel-move based surfaces. We have obtained exact six types of regular inner surface points [8]. In practice, it is may useful to extract all simple surface points in a set S by matching the six types of simple surface points. Theoretically, to decide if S is a
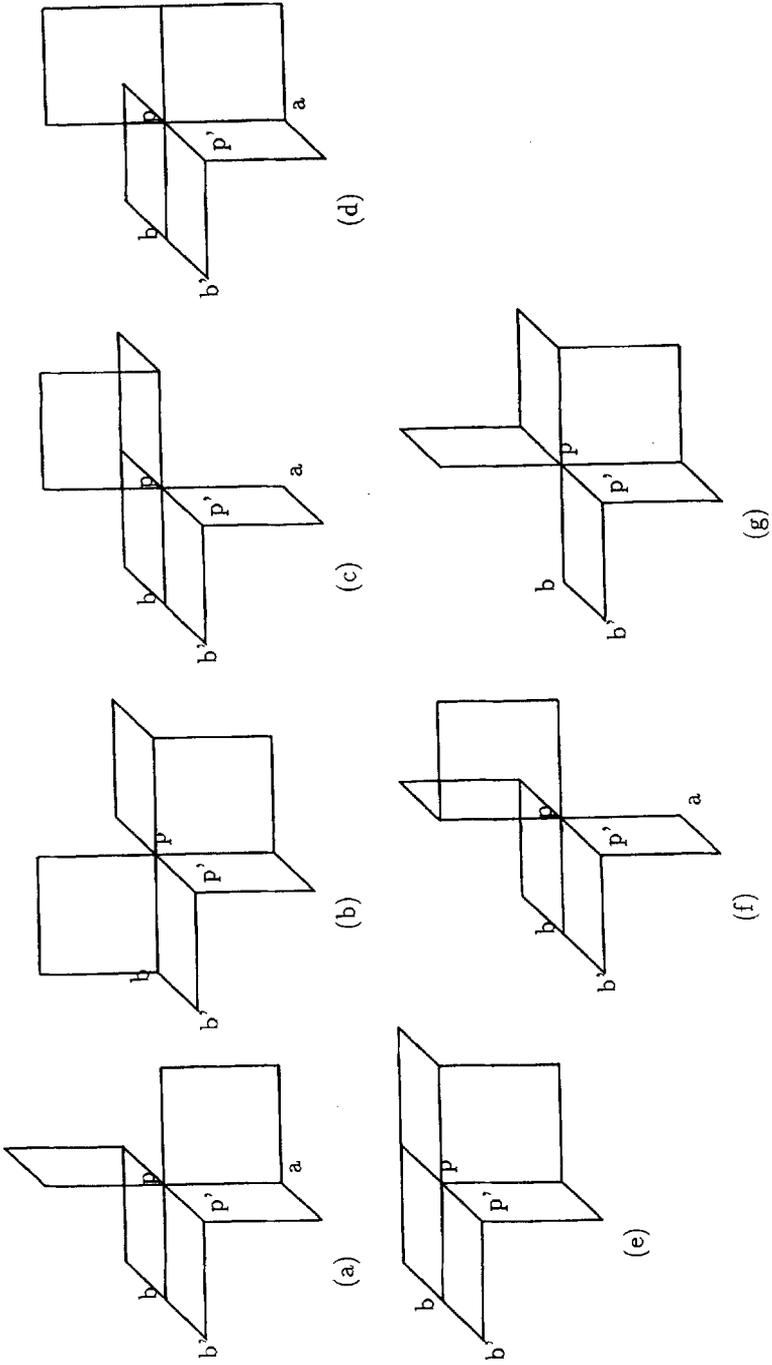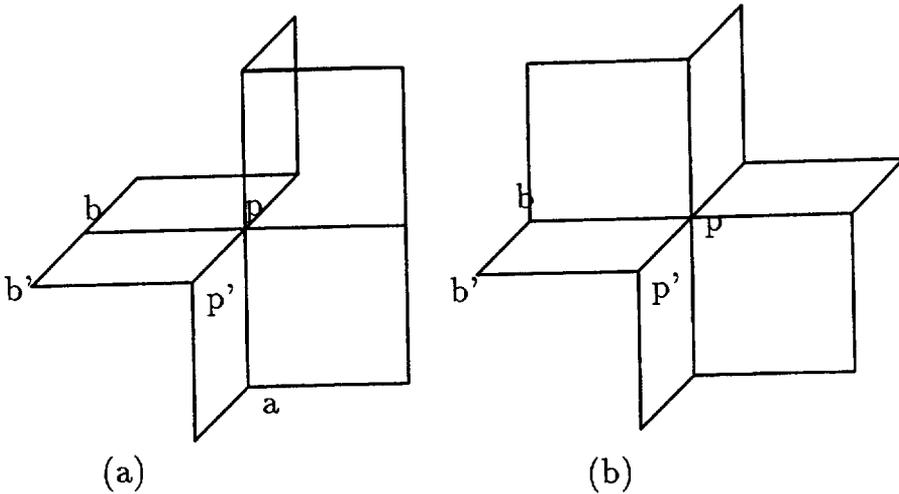
Fig. 15. Seven cases.

Fig. 16. Two cases.

simple surface need not to test all three conditions of definition 2.1 or to match six types of regular inner surface points.

A Morgenthaler–Rosenfeld's simple surface S is closed, and each point is a simple surface point. A fast algorithm can be developed by testing if S is a parallel-move based closed surface first, and then testing if S contains the case of Fig. 7(b).

In addition, the definition of simple surface points cannot deal with boundary points. The drawback of the definition of regular surface points is how to expand it to deal with indirect adjacency. We discuss this issue in [7]. Comparing with Kong and Roscoe's results [6], Definition 4.1 has some similarities to Proposition 6 in [6]. However, Definition 4.1 is much simpler, intuitive and general.

In summary, this paper proved that Morgenthaler–Rosenfeld's simple surfaces are just cases of regular closed surfaces defined by the authors in [2,3] in direct adjacency. We also can see there are only six types of simple surface points in three dimensional grid spaces [8].

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